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# The Paley–Wiener Theorem for the Hua System

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E. Damek, A. Hulanicki, and R. Penney (*J. Funct. Anal.*, in press) studied a canonical system of differential equations (the Hua system) denoted $HJK$ which is definable on any Kählerian manifold $M$. Functions annihilated by this system are called "Hua-harmonic." In the case where $M$ is a bounded homogeneous domain in $\mathbf{C}^n$ with its Bergman metric, it was shown that every bounded Hua-harmonic function has a boundary value on the Bergman–Shilov boundary and that the function is reproducible from the Shilov boundary by integration against the reduction of the Poisson kernel for the Laplace–Beltrami operator to the Shilov boundary. This then provided a partial generalization of the results of Johnson and Korányi to the stated context. Significantly, however, no characterization of the resulting space of boundary functions so obtained was given. The current work extends these results in several ways. We show that for a tube domain (i.e., a Siegel domain of type I), the Cauchy–Szegö Poisson kernel also reproduces the Hua-harmonic functions. Since the two kernels agree only in the symmetric case, it follows that the space of boundary functions is dense in $L^\infty$ if and only if the domain is symmetric. We also show that an $L^2$ function is the boundary function for a Hua-harmonic function if and only if its Fourier transform is supported in a certain (typically non-convex) cone. This cone is characterized in terms of the Fourier transformation of the Cauchy–Szegö Poisson kernel. 

## 1. INTRODUCTION

One of the more beautiful results in the harmonic analysis of symmetric spaces is the Helgason Theorem, which states that on a Riemannian symmetric space $X = G/K$, a function is annihilated by the algebra $D_G(X)$ of all $G$-invariant differential operators if and only if it is the Poisson integral of a hyperfunction over the "maximal" boundary. (See [KKMOOT].)

If $X$ is a Hermitian symmetric space, then one is typically interested in complex function theory, in which case one is interested in functions whose boundary values are supported on the Shilov boundary rather than the maximal boundary. In this case, it turns out that the algebra of $G$ invariant



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differential operators is not necessarily the most appropriate one for defining harmonicity. Johnson and Koryáni [JK], generalizing earlier work of Hua [Hu], Koryáni and Stein [KS], and Koryáni and Malliavin [KM], introduced an invariant system of second order differential operators (the *HJK* system) defined on any Hermitian symmetric space. They showed that any function that is annihilated by this system (i.e., any *Hua-harmonic* function) is the Poisson integral of a hyperfunction over the Shilov boundary. They also showed that in the special case that $X$ is a tube domain, all Poisson integrals are Hua-harmonic. Thus, in the tube case, the Hua system plays the same role with respect to the Shilov boundary as the algebra $D_G(X)$ does with respect to the maximal boundary. (Later, Lassalle ([La1, La2]) showed the existence of a smaller *real* system with the same properties as the Johnson–Koryáni system. This smaller system will not, however, play a role in the current work.)

In the general Hermitian symmetric case, it is not true that all Poisson integrals are Hua-harmonic. In [BV], Berline and Vergne commented that the boundary values of the Hua-harmonic functions should satisfy some "tangential" Hua equations. They also produced an invariant system of third order operators with the property that a function $f$ is the Poisson integral of a hyperfunction over the Shilov boundary if and only if $f$ is annihilated both by the Berline-Vergne system and by $D_G(X)$. Arguably, however, the Hua system is perhaps more appropriate for the study of analytic function theory than the Berline–Vergne system, since it is simplier and defines a smaller class of boundary functions. (Both systems annihilate holomorphic functions.)

Every Hermitian symmetric space is, of course, a Kähler manifold. In [DHP2] it was noted that the *HJK* system is definable on any Kähler manifold $X$. This more general system is invariant under any biholomorphic isometry of the manifold. It seems interesting to ask to what extent the results of Johnson and Koryáni depend on the semi-simplicity of the space and to what extent they are special cases of results valid for a larger class of Kählerian manifolds. Specifically, one is interested in the following questions.

(1) Given a Kählerian manifold $X$, is there a Poisson kernel on the Shilov boundary for $X$ with the property that every function which is annihilated by the *HJK* system on $X$ is the Poisson integral of a hyperfunction over the Shilov boundary?

(2) If the answer to the first question is affirmative, can we describe the space of boundary functions for *HJK*?

In the light of the Helgason theorem, it is natural to restrict initially to *homogeneous* Kähler manifolds. Then a result of Dorfmeister and Nakajima

[DN] states that the general such manifold decomposes as a fiber bundle over a bounded homogeneous domain in $\mathbf{C}^n$ where the fibers are homogeneous Kähler manifolds of a particularly simple type. Thus, it is natural to restrict further to the class of bounded homogeneous domains in $\mathbf{C}^n$. Note that this class still contains all Hermitian symmetric manifolds.

Question (1) was studied in [DHP2] where it was shown that in the bounded-homogeneous case there is indeed a "Poisson" kernel on the Shilov boundary that reproduces the Hua-harmonic functions. In fact, it was shown that the Shilov boundary is a boundary (in the sense of [DH]) for the Laplace–Beltrami operator of the domain and that the Poisson kernel for this operator on the Shilov boundary suffices to reproduce the Hua-harmonic functions. It should be noted that the Laplace–Beltrami operator is a linear combination of operators from the Hua system so the Hua-harmonic functions are, in particular, harmonic for the Laplace–Beltrami operator. Typically, the maximal boundary for this operator is larger than the Shilov boundary ([DHP1]). Thus, the main content of the theorem for the *HJK* system just mentioned is that the boundary values for the *HJK* system, which initially exist only on the maximal boundary, are actually supported on the (smaller) Shilov boundary.

In the case of a symmetric domain, the Poisson kernel for the Laplace–Beltrami operator is easily computable in terms of the complex structure of the domain. Specifically, let $S(z, w)$ be the Szegö kernel function for the domain. (This is the reproducing kernel for $\mathscr{H}^2$.) Then, in this case, $S$ extends almost everywhere in $w$ to the Shilov boundary and the function

$$P(z, x) = \frac{|S(z, x)|^2}{S(z, z)}$$

where $z$ belongs to the domain and $x$ to the Shilov boundary, is the Poisson kernel for the Laplace–Beltrami operator. This function is called the *Cauchy–Szegö Poisson kernel.*

For a non-symmetric domain, the Cauchy–Szegö Poisson kernel is not the Poisson kernel for the Laplace–Beltrami operator. In fact, it is known that the Cauchy–Szegö Poisson kernel is harmonic for the Laplace–Beltrami operator if and only if the domain is symmetric [Xu]. There is, to our knowledge, no general formula for the Laplace–Beltrami kernel outside of the symmetric case. This then tends to diminish the utility of the result mentioned above concerning the reproducibility of the Hua-harmonic functions from the boundary.

The first main result of this work is the remarkable statement that *the Cauchy–Szegö Poisson kernel also reproduces Hua-harmonic functions.* Thus, the two most natural candidates for a Poisson kernel, the Cauchy–Szegö Poisson kernel and the Laplace–Beltrami Poisson kernel, *both work* *equally well for the Hua-harmonic functions*. This is all the more remarkable when one realizes that in the non-symmetric case, the Hua system does not annihilate the Cauchy–Szegö Poisson kernel. (Recall that the Laplace–Beltrami operator is a linear combination of operators from the Hua system.) Thus, there is no a priori reason to expect a connection between the Hua system and the Cauchy–Szegö Poisson kernel. It should also be noted that there is a considerable body of information relating to the Cauchy–Szegö Poisson kernel. (See, for example, [DHP1].)

The non-uniqueness of the reproducing kernel of course means that the space of boundary values of the Hua-harmonic functions cannot be dense in $L^\infty$ of the boundary. Thus, a complete understanding of the Hua-harmonic functions requires describing the space formed by their boundary values. The second major result of this work is a characterization of the space of $L^2$ boundary values. To describe this result we must recall the definition of the homogeneous Siegel domains of type I. It should be noted that every symmetric tube domain has a realization as a Siegel domain of type I.

Let $M$ be a finite dimensional real vector space and let $\mathscr{V} \subset M$ be an open, convex cone that does not contain straight lines. (Such cones are said to be *regular*.) Then the Siegel domain of type I defined by $\mathscr{V}$ is the domain $\mathscr{D} \subset M_c$,

$$\mathscr{D} = M + i\mathscr{V}.$$

It is known that $\mathscr{D}$ is biholomorphically equivalent to a bounded domain in $\mathbf{C}^n$.

We assume that the cone $\mathscr{V}$ is homogeneous, i.e., there is a real algebraic group $S$, an algebraic representation $\rho$ of $S$ on $M$, and a point $c \in \mathscr{V}$ for which $\mathscr{V} = \rho(S)\,c$. It is well known that in this case $S$ may be chosen to be completely solvable and to act simply transitively on $\mathscr{V}$ [Vin]. We shall assume that $S$ has been so chosen.

Under these assumptions, $S$ acts on $\mathscr{D}$ by means of $\rho$. The group $M$ also acts on $\mathscr{D}$ by translation. In fact $\mathscr{D}$ is homogeneous under the semi-direct product $G = M \times_s S$ where $S$ acts on $M$ by means of $\rho$. This action makes $\mathscr{D}$ into a *homogeneous* Siegel domain of type I.

The set $M = M + 0i$ is referred to as the "Bergman–Shilov" boundary of $\mathscr{D}$—it is an open dense subset of the Shilov boundary. The second main result of this work is the statement that *a function in* $L^2(M, dx)$, *where* $dx$ *is Lebesgue measure, is the boundary value of a Hua-harmonic function if and only if its Fourier transformation is supported in a certain open subset* $\mathcal{O} \in M^*$. *This set is invariant under the contragradient action of* $S$ *on* $M^*$ *and is a finite union of open* $S$ *orbits*. Thus, describing the space of boundary functions comes down to determining which of the $S$ orbits in $M^*$ are contained in $\mathcal{O}$. Such orbits are said to be *harmonic*. It should be noted that the $S$ orbits are cones that are typically non-convex.

We think of this result as an analogue of the classical Paley–Wiener theorem in that it describes the space of boundary values solely in terms of the support of their Fourier transformations. It is also analogous to results of Rossi and Vergne [RV] relating to the tangential Cauchy–Riemann equations. In fact, it occurs for much the same reasons as in [RV].

Determining the harmonic orbits is an important and, as yet, unsolved problem. However, there is a result that has some promise of yielding significant insight into this issue. To describe this result, let $P(z, x)$ denote the Cauchy–Szegö Poisson kernel for $\mathcal{D}$, where now $z$ ranges over $\mathcal{D}$ and $x$ ranges over $M$. For $\lambda \in M^*$ let

$$P\,\hat{}\,(z, \lambda) = \int_{-\infty}^{\infty} P(z, x)\, e^{-i\langle x, \lambda\rangle}\, dx$$

be the Fourier transformation of $P$ in the $x$ variable. We show that if $\lambda$ belongs to an open orbit of $S$, then *the corresponding orbits is harmonic if and only if* $P\,\hat{}\,(z, \lambda)$ *is Hua-harmonic as a function of* $z$.

## 2. HOMOGENEOUS CONES

We continue the notation defined in the introduction. Specifically, we assume that $M$, $S$, $\rho$, $c$, and $\mathcal{V}$ are as defined at the end of the introduction. The 4-tuple $(S, M, c, \rho)$ is referred to as "tube data." The following example plays an important role in this work.

EXAMPLE (1.1). Let $M^n$ be the space of all $n \times n$ real, symmetric matrices and let $\mathcal{V}^n$ be the cone of all positive definite elements of $M^n$. Let $S^n$ be the group of $n \times n$ upper triangular matrices with positive diagonal. For $s \in S^n$ and $X \in M^n$, we define

$$\rho^n(s)\, X = sXs^t$$

where $s^t$ is the transpose of $s$. Then, as is well known, $(S^n, M^n, I, \rho^n)$ is tube data for a Hermitian symmetric tube domain.

It is classical that the domain $\mathcal{D}$ is biholomorphically equivalent with a bounded domain. As such, it has a canonical Riemannian structure defined from the Bergman metric. Since $G$ acts simply transitively on $\mathcal{D}$, the tangent space at $ic$ may be identified with the Lie algebra $\mathcal{G}$ of $G$.

In general, *we adopt the convention that Lie groups are denoted by upper case Roman letters and the corresponding Lie algebras are denotes by the corresponding upper case script letter.*

Since the Riemannian structure is $G$-invariant, it is defined by a scalar product $g$ on the Lie algebra $\mathscr{G}$. Koszul [Kl, Formula 4.5] proved the existence of a functional $\beta \in \mathscr{G}^*$ such that this scalar product is given by

$$g(X, Y) = \beta([JX, Y]), \tag{1}$$

where $J: \mathscr{G} \to \mathscr{G}$ defines the complex structure on $\mathscr{G}$. We shall not explicitly use any other information concerning $\beta$ other than the fact that formula (1) defines a $J$-invariant, positive-definite, scalar product.

More explicitly, let $\mathscr{M}$ and $\mathscr{S}$ be the respective Lie algebras for $S$ and $M$. Of course, since $M$ is a vector space, we may identify $\mathscr{M}$ and $M$. The representation $\rho$ defines a Lie algebra representation (also denoted $\rho$) of $\mathscr{S}$ on $\mathscr{M}$. Then $\mathscr{G}$ is $\mathscr{M} \times_s \mathscr{S}$ where the semi-direct product is defined from $\rho$. Since $S$ acts simply transitively on $\mathscr{V}$, the mapping $X \to \rho(X)\, c$ defines a vector space isomorphism of $\mathscr{S}$ onto $\mathscr{M}$. In [DHP2], it is shown that there is s functional $\xi \in \mathscr{M}^*$ such that

$$g(\rho(X_1)\, c \times Y_1, \rho(X_2)\, c \times Y_2) = \xi(\rho(X_2)\, \rho(X_1)\, c) + \xi(\rho(Y_2)\, \rho(Y_1)\, c). \tag{2}$$

(See Lemma (2.5) in [DHP2].)

*Note that we denote the general element of a product space* $X \times Y$ *as* $x \times y$ *rather than the more common* $(x, y)$.

Now let $(S_1, M_1, c_1, \rho_1)$ and $(S_2, M_2, c_2, \rho_2)$ be two sets of tube data. Then a homomorphisim from the first set of tube data to the second is a pair $(\tau, T)$ consisting of a homomorphism $\tau: S_1 \to S_2$ and a mapping $T: M_1 \to M_2$ such that

(a) $T(c_1) = c_2$;

(b) for all $s \in S_1$,

$$\rho_2(\tau(s))\, T = T\rho_1(s)$$

It follows that $T(\mathscr{V}_1) \subset \mathscr{V}_2$.

A homomorphism of a given set of tube data into the tube data of Example (1.1) is said to be a *representation* of the tube data in $\mathbf{R}^n$. Specifically, a representation of $(S, M, c, \rho)$ is a pair $(\tau, T)$, where $\tau$ is a representation of $S$ by $n \times n$ upper triangular matrices and $T$ is a mapping of $M$ into the space of $n \times n$ symmetric matrices where

(a) $T(c) = I$ where $I$ is the $n \times n$ identity matrix.

(b) For all $s \in S$ and $m \in M$,

$$T(\rho(s)\, m)) = \tau(s)\, T(m)\, \tau(s)^t.$$

Note that it follows that $T$ maps $\mathscr{V}$ into the cone of positive definite matrices.

Representations are important in part because they provide an inductive procedure (due to Rothaus [Ro]) for constructing cones. To explain this, let $(\tau_o, T_o)$ be a representation of $(S_o, M_o, c_o, \rho_o)$ in $\mathbf{R}^n$. Let $S$ be the set of all matrices $s$ of the form

$$s = \begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix}, \tag{3}$$

where $a \in \mathbf{R}^+$, $v \in \mathbf{R}^n$ (thought of as column vectors), $s_o \in S_o$, and 0 is the zero element of $\mathbf{R}^n$. We define the product of two such elements via

$$\begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix} \begin{bmatrix} b & u^t \\ 0 & t_o \end{bmatrix} = \begin{bmatrix} ab & au^t + [\tau_o(t_o)\, v]^t \\ 0 & s_o t_o \end{bmatrix}.$$

It is easily seen that $S$ becomes a Lie group under this action.

Next let $M$ be the vector space of all matrices of the form

$$m = \begin{bmatrix} b & w^t \\ w & m_o \end{bmatrix}, \tag{4}$$

where $b \in \mathbf{R}$, $w \in \mathbf{R}^n$, and $m_o \in M_o$. For $s$ as in formula (3) and $m$ as above, we define

$$\rho(s)\, m = \begin{bmatrix} a^2 b + 2a(v, w) + (T_o(m_o)\, v, v) & [\tau_o(s)(aw + T_o(m_o)\, v)]^t \\ \tau_o(s)(aw + T_o(m_o)\, v) & \rho_o(s)\, m_o \end{bmatrix}, \tag{5}$$

where $(\cdot,\ \cdot)$ is the Euclidian scalar product on $\mathbf{R}^n$. It is easily seen that $\rho$ is a representation of $S$ on $M$. (This formula results from formally expanding the matrix product

$$\begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix} \begin{bmatrix} b & w^t \\ w & m_o \end{bmatrix} \begin{bmatrix} a & 0 \\ v & s_o^t \end{bmatrix}$$

using $(\cdot,\ \cdot)$, $\tau_o$, and $T_o$ to define the products of individual elements, where we interpret $s_o m_o s_o^t$ as $\rho_o(s_o)\, m_o$.)

Finally, let $c_o \in \mathscr{V}_o$ and set

$$c = \begin{bmatrix} 1 & 0 \\ 0 & c_o \end{bmatrix}.$$

Let $\mathscr{V}$ be the $S$ orbit of $c$ in $M$. It is a result of Rothaus that $\mathscr{V}$ is an open, regular, convex cone which is independent of the choice of $c_o$ in $\mathscr{V}_o$. We refer to $(S, M, c, \rho)$ as the cone data induced from $(S_o, M_o, c_o, \rho_o)$ using the representation $(\tau_o, T_o)$. Rothaus also showed that *every homogeneous cone is isomorphic to one induced from a lower dimensional cone using an appropriate representation.*

As an example of this construction, we note that in Example 1.1, the usual action of $S^n$ on $\mathbf{R}^n$ defines a representation $\tau^n$ of $S^n$. The elements of $M^n$ are symmetric matrices so the identity transformation defines a mapping to $M^n$ into the space of symmetric matrices. The pair $(\tau^n, I)$ is a representation of $(S^n, M^n, I, \rho^n)$. Then, as the reader may easily verify, the corresponding induced cone data is just $(S^{n+1}, M^{n+1}, I, \rho^{n+1})$.

The Lie algebra $\mathscr{G}$ is easily described. As a vector space $\mathscr{G} = \mathscr{M} \times \mathscr{S}$ where $\mathscr{M}$ and $\mathscr{S}$ are, respectively, the Lie algebras of $M$ and $S$. Of course $\mathscr{M} = M$ since $M$ is a vector space. The space $\mathscr{S}$ is the set of matrices $s$ of the form

$$s = \begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix},$$

where $s_o \in \mathscr{S}_o$.

As a Lie algebra, $\mathscr{G} = \mathscr{M} \times_s \mathscr{S}$, where the action of $s \in \mathscr{S}$ on $\mathscr{M}$ is found by differentiating formula (5) in the direction of $s$ at the identity. It is given by

$$\rho(s) \begin{bmatrix} b & w^t \\ w & m \end{bmatrix} = \begin{bmatrix} 2ab + 2(v, w) & [aw + T_o(m_o)\, v + \tau_o(s_o)\, w]^t \\ aw + T_o(m_o)\, v + \tau_o(s_o)\, w & \rho_o(s)\, m_o \end{bmatrix},$$

where $\tau_o$ and $\rho_o$ are, respectively, the Lie algebra representations obtained by differentiating $\tau_o$ and $\rho_o$.

We shall also require a description of the scalar product on the induced cone. For this we note the following well known result. We sketch the proof for sake of completeness.

LEMMA 2.1. *Let the cone data $(S, M, c, \rho)$ be induced as described above. Then the functional $\xi$ from formula (2) is zero on all elements of $\mathscr{M}$ of the form*

$$\begin{bmatrix} 0 & w^t \\ w & 0 \end{bmatrix}.$$

*Proof.* This follows very simply from the following formula which is a consequence of the symmetry of the scalar product (we leave the details to the reader):

$$\xi(\rho([X_2, X_1])\, c) = \xi(\rho(X_2)\, \rho(X_1)\, c) - \xi(\rho(X_1)\, \rho(X_2)\, c) = 0.$$

As a direct consequence, we have the following:

LEMMA 2.2. *Let $X \in \mathcal{G}$ with $X = m \times s$, where*

$$m = \begin{bmatrix} b & w^t \\ w & 0 \end{bmatrix} \qquad s = \begin{bmatrix} a & v^t \\ 0 & 0 \end{bmatrix}.$$

*Let*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

*Then*

$$g(X, X) = 2\xi(E)(2a^2 + (v, v) + b^2/2 + (w, w)).$$

From now on, we will assume that $(S, M, c, \rho)$ is induced as described above. There are a number of subgroups of $G$ which play an important role. Specifically, we define the named set on the left in the list below to be the set of all elements of $S$ of the form described on the right, where $e_o$ is the identity element of $S_o$, $s_o$ ranges over $S_o$, $v$ and $w$ range over $\mathbf{R}^n$, $a$ ranges over $\mathbf{R}^+$, and $b$ ranges over $\mathbf{R}$:

$S_H$ typical element: $\begin{bmatrix} a & v^t \\ 0 & e_o \end{bmatrix}$

$A_H$ typical element: $\begin{bmatrix} a & 0 \\ 0 & e_o \end{bmatrix}$

$N_H$ typical element: $\begin{bmatrix} 1 & v^t \\ 0 & e_o \end{bmatrix}$

$M_H$ typical element: $\begin{bmatrix} b & w^t \\ w & 0 \end{bmatrix}$

$M_H^o$ typical element: $\begin{bmatrix} 0 & w^t \\ w & 0 \end{bmatrix}$.

We also identify the groups $S_o$ and $M_o$ respectively with the subgroups of $S$ and $M$ described below:

$$S_o \qquad \text{typical element:} \qquad \begin{bmatrix} 1 & 0 \\ 0 & s_o \end{bmatrix}$$

$$M_o \qquad \text{typical element:} \qquad \begin{bmatrix} 0 & 0 \\ 0 & m_o \end{bmatrix}.$$

Finally, we define the following subgroups of $G = M \times_s S$:

$$G_o = M_o S_o$$

$$G_H = M_H S_H$$

$$H = M_H N_H.$$

It is easily seen that $H$ is a normal subgroup of $G$. The reason for calling this subgroup $H$ is that it is a Heisenberg group—i.e., it is a two step nilpotent Lie group with one-dimensional center. In fact, its center is the set of elements in $M$ such that $w = 0$ and $m_o = 0$.

The orbit of $ic$ in $M_c$ under $G_H$ is the set of elements of $M$ of the form

$$\begin{bmatrix} b + i(a^2 + |v|^2) & w^t + iv^t \\ w + iv & ic_o \end{bmatrix}.$$

This set is identifiable with the domain $\mathscr{B}$ in $\mathbf{C} \times \mathbf{C}^n$ consisting of all points $(W, Z)$ such that im $W > |\mathrm{im}(Z)|^2$.

The domain $\mathscr{B}$ is, in fact, equivalent with the unit ball in $\mathbf{C}^{n+1}$. The simplest way to prove this is to note that the transformation

$$(W, Z) \to \left(2W - i \sum Z_i^2, Z\right)$$

transforms $\mathscr{B}$ into the domain described by im $W > |Z|^2$, which is well known to be equivalent to the unit ball. Since $G_H$ acts simply transitively on $\mathscr{B}$, we may identify $G_H$ with the unit ball in $\mathbf{C}^{n+1}$.

There is a representation of $(S, M, c, \rho)$ that plays an important role. For $s$ as in formula (3), we define $\tau(s)$ to be the operator on $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n$ defined by the matrix

$$\begin{bmatrix} a & v^t \\ 0 & \tau_o(s_o) \end{bmatrix}.$$

Similarly, for $m$ as in formula (4), we define $T(m)$ to be the operator on $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n$ defined by the matrix

$$\begin{bmatrix} b & w^t \\ w & T_o(m_o) \end{bmatrix}.$$

It is easily seen that $(\tau, T)$ is a representation of $(S, M, c, \rho)$ on $\mathbf{R}^{n+1}$. We refer to this representation as the representation induced from the representation $(\tau_o, T_o)$ of $(S_o, M_o, c_o, \rho_o)$. Notice that the mapping

$$T \times \tau : M \times_s S \to M^n \times_s S^n$$

is a group homomorphism that restricts to an isomorphism of $G_H$ onto $G_H^n$. Lemma (2.2) tells us that the corresponding Lie algebra isomorphism is a scalar multiple of an isometry of $\mathcal{G}_H$ onto $\mathcal{G}_H^n$.

If $\mathscr{V}$ is a regular cone in a vector space $\mathscr{M}$, then we define the dual cone $\mathscr{V}^*$ be the set of elements $\lambda \in \mathscr{M}^*$ that are strictly positive on $\bar{\mathscr{V}} - 0$. The group $S$ acts on $\mathscr{V}^*$ via the adjoint representation which is defined by the formula

$$\rho^*(s) = \rho(s^{-1})^*.$$

It is known that $\mathscr{V}^*$ is homogeneous under this action.

For each $m \in \mathscr{V}$, we define

$$D(m) = \int_{\mathscr{V}^*} e^{-\langle m, \lambda \rangle}\, d\lambda \tag{6}$$

where $d\lambda$ denotes Lebesgue measure on $\mathscr{V}^*$, *which we normalize so that* $D(c) = 1$. This function is referred to as the *characteristic function* for the cone. The absolute convergence of this integral for all $v \in \mathscr{V}$ is proved in Vindberg [V].

We would like to describe $D$ inductively. To this end, we note that a simple change of variables in formula (6) shows that for all $s \in \mathscr{S}$ and all $v \in \mathscr{V}$,

$$D(\rho(s)\, m) = \chi(s)^{-1}\, D(m), \tag{7}$$

where

$$\chi(s) = \det(\rho(s)).$$

Since $D(c) = 1$, it follows that $D$ maps $\rho(s)\, c$ into $\chi(s)^{-1}$.

For $s_o \in S_o$, let

$$\chi_o(s_o) = \det(\rho_o(s_o)) \qquad \text{and} \qquad \kappa(s_o) = \det(\tau_o(s_o)).$$

LEMMA 2.3. *For $s$ as in formula* (3),

$$\chi(s) = a^{n+2} \kappa(s_o)\, \chi_o(s_o).$$

*Proof.* We note that

$$\begin{bmatrix} a & v^t \\ 0 & s_o \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & s_o \end{bmatrix} \begin{bmatrix} 1 & a^{-1}v^t \\ 0 & e_o \end{bmatrix}.$$

Let us call the first matrix on the right $\delta$ and the second $u$. Since $u$ is unipotent, $\chi(u) = 1$. Hence $\chi(s) = \chi(\delta)$. Our lemma follows easily by taking $v = 0$ in formula (5).

Now, suppose that $m = \rho(s)\, c$. Then, from formula (5), $m_o = \rho_o(s_o)\, c_o$. Hence

$$\chi_o(s_o) = D_o(m_o)^{-1},$$

where $D_o$ is the characteristic function for $\mathscr{V}_o$. Furthermore,

$$T_o(m_o) = \tau_o(s_o)\; T_o(c_o)\; \tau_o(s_o)^t = \tau_o(s_o)\; \tau_o(s_o)^t.$$

Finally, we note that a similar argument proves that

$$\det(T(m)) = \det(\tau(s))^2 = a^2 \det(\tau_o(s_o))^2 = a^2 \det(T_o(m_o)).$$

Thus, substitution into the formula from Lemma (2.3) yields

$$\begin{aligned} D(m) &= \chi(m)^{-1} \\ &= \left( \frac{\det(T(m))}{\det(T_o(m_o))} \right)^{-(n+2)/2} \det(T_o(m_o))^{-1/2}\, D_o(m_o) \\ &= \det(T(m))^{-(n+2)/2} \det(T_o(m_o))^{(n+1)/2}\, D(m_o). \end{aligned}$$

The following (well known) corollary follows from the above formula by induction. (See [KF, p. 11].)

Corollary 2.4. *Let $D^n$ be the characteristic function for the cone $\mathscr{V}_n$ of Example* (1.1). *Then for all $m \in \mathscr{V}_n$,*

$$D^n(m) = \det(m)^{-(n+1)/2}.$$

We note the following consequence of this corollary which will be used in the next section:

$$\frac{D(m)}{D(m_o)} = \frac{D^{n+1}(T(m))}{D^n(T_o(m_o))}. \tag{8}$$

## 3. THE POISSON KERNEL FOR THE HJK SYSTEM

We continue the notation established in Section 2. Specifically, we assume that $\mathscr{D}$ is the tube domain defined by the cone data $(S, M, c, \rho)$ which is induced from the representation $(\tau_o, T_o)$ of $(S_o, M_o, c_o, \rho_o)$.

Our goal in this section is to prove that a bounded Hua-harmonic function $F$ is reproducible from its boundary value function by integration against the Cauchy–Szegö Poisson kernel. Our proof will rely heavily on one of the main results of [DHP2]—namely that there is a Poisson kernel on the Bergman–Shilov boundary that reproduces Hua harmonic functions from their boundary values. Actually, in [DHP2] we proved for $F$ to be reproducible using the stated kernel, it sufficed that $F$ be harmonic for a smaller system, called the *strongly diagonal Hua system*. Functions harmonic for this system are referred to as *diagonally harmonic*. It is this stronger result that we use. We will not need to recall the definitions of either the Hua system or of the strongly diagonal Hua operators since we will only require a few of their general properties from [DHP2].

Now, let $F$ be a bounded, diagonally harmonic function on $G = M \times_s S$. In [DHP2], Theorem (2.18), we described one particular strongly diagonal operator denoted $HJK_1$. This operator had the form

$$HJK_1 = \xi(E)^{-2} \left[ 2(Y^2 + X^2) - (n+2)\, Y + \sum_1^n Y_j^2 + X_j^2 \right],$$

where $E$ and $\xi$ are as in Lemma 2.2, $\xi(E)^{-1/2}\, Y_i$ is an orthonormal basis for $\mathscr{N}_H$, $\xi(E)^{-1/2}\, X_i$ is an orthonormal basis for $\mathscr{M}_H^o$, and $Y = 0 \times E/2$ and $X = E \times 0$. (Note that under the obvious identifications of $\mathscr{N}_H$ and $\mathscr{M}_H^o$ with $\mathbf{R}^n$, $X_i$ and $Y_i$ are orthonormal bases with respect to $2(\cdot, \cdot)$. Thus, $HJK_1$ is independent of the choice of $\xi$, up to scalar multiples.)

Note that $HJK_1$ is defined by an element $\Delta_H$ in the enveloping algebra of $G_H$. In fact, it is easily seen that $\Delta_H$ is just the Laplace–Beltrami operator for the unit ball in $\mathbf{C}^{n+1}$ under the identification of $G_H$ with the unit ball described in Section 2. It follows from [DHP1] that the maximal boundary for $\Delta_H$ on $G_H$ is $H$, which we identify with $G_H/A_H$. Let $P_H$ be the Poisson kernel function for $\Delta_H$ on $H$. Since $F \mid G_H$ is $\Delta_H$-harmonic, there is a function $f_H$ on $G_H$ which is constant on cosets of $A_H$ in $G_H$ such that

$$F(e) = \int_H f_H(h)\, P_H(h)\, dh. \tag{9}$$

Let $\delta(t)$ be the element of $G$ defined by

$$\delta(t) = 0 \times \begin{bmatrix} t & 0 \\ 0 & I \end{bmatrix}.$$

LEMMA 3.1. *Let notation be as above. Assume that $f_H$ is continuous on $H$. Then for all $k \in H$,*

$$\lim_{t \to 0} F(k\delta(t)) = f_H(k).$$

*Proof.* For all $g \in G$, we set $g(t) = \delta(t)\, g\delta(t)^{-1}$. Thus, if

$$h = \begin{bmatrix} b & w^t \\ w & 0 \end{bmatrix} \times \begin{bmatrix} 1 & v^t \\ 0 & I \end{bmatrix}$$

then

$$h(t) = \begin{bmatrix} t^2 b & tw^t \\ tw & 0 \end{bmatrix} \times \begin{bmatrix} 1 & tv^t \\ 0 & I \end{bmatrix}.$$

As $t \to 0$, $h(t)$ ends to the identity element. Then, for all $k \in H$,

$$F(k\delta(t)) = \int_H f_H(k\delta(t)\, h)\, P_H(h)\, dh$$

$$= \int_H f_H(kh(t))\, P_H(h)\, dh,$$

where $f$ is the boundary function of $F$.

The function $P$ may not be unique. However, we assume that one specific $P$ has been chosen. From the above lemma, for all $k \in H$,

$$\begin{aligned} f_H(k) &= \lim_{t \to 0} F(k\delta(t)) \\ &= \lim_{t \to 0} \int_M f(k\delta(t)\, m)\, P(m)\, dm \\ &= \lim_{t \to 0} \int_M f(km(t))\, P(m)\, dm. \end{aligned}$$

Now, for $m \in M$, we may write $m = m_H + m_o$, where $m_H \in M_H$ and $m_o \in M_o$. Then, since $\delta(t)$ centralizes $M_o$,

$$m(t) = m_H(t) + m_o.$$

Noting that $m_H(t)$ tends to 0 as $t \to 0$, we see that

$$f_H(k) = \int_{M_o} f(km_o)\, P_o(m_o)\, dm_o, \tag{10}$$

where

$$P_o(m_o) = \int_{M_H} P(m_o + m_H)\, dm_H. \tag{11}$$

Putting formulas (9) and (10) together, we see that

$$F(e) = \int_H \int_{M_o} f(km_o)\, P_H(k)\, P_o(m_o)\, dm_o\, dk. \tag{12}$$

The function $P_o$ has an important interpretation. As commented earlier, the group $G_o$ acts transitively on the domain $\mathscr{D}_o$. Let $HJK_o$ be the corresponding Hua system. From Lemma (2.21) in [DHP2], a function $F_o$ on $G_o$ is diagonally harmonic for $HJK_o$, if and only if it is the restriction to $G_o$ of an $HJK$ diagonally harmonic function $\tilde{F}$ that is constant on $G_H$ cosets in $G$. The boundary function $\tilde{f}$ of $\tilde{F}$ will be constant on cosets of $M_H$ in $M$. But then

$$\begin{aligned} \tilde{F}(e) &= \int_M \tilde{f}(m)\, P(m)\, dm \\ &= \int_{M_H} \int_{M_o} \tilde{f}(m_H + m_o)\, P(m_H + m_o)\, dm_H\, dm_o \\ &= \int_{M_o} \tilde{f}(m_o)\, P_o(m_o)\, dm_o. \end{aligned}$$

It follows that the restriction of $P_o$ to $M_o$ is a Poisson kernel function for the diagonally harmonic functions on $G_o$. Actually, *in formula* (12) *we may replace* $P_o$ *by any Poisson kernel function for the strongly diagonal Hua operators on* $G_o$. To see this, it suffices to show that for all $k \in H$, the function

$$m_o \to f(km_o)$$

is the boundary function for a diagonally harmonic function on $G_o$, since then the integral in formula (10) will be independent of the particular kernel chosen. Since our differential operators commute with left translation, it in fact suffices to assume that $k = e$.

However, for $g \in G$, let

$$F_o(g) = \lim_{t \to 0} F(\delta(t)\ g).$$

LEMMA 3.2. *The limit defining* $F_o$ *exists for all* $g \in G$ *and defines a diagonally harmonic function that is constant on cosets of* $G_H$ *on* $G$. *The corresponding boundary function equals* $f$ *on* $M_o$.

*Proof.* Let $g \in G$. We may write

$$g = g_o g_H,$$

where $g_H \in G_H$ and $g_o \in G_o$. Then,

$$\begin{aligned} F_o(g) &= \lim_{t \to 0} \int_M f(\delta(t)\ g_o g_H m)\ P(m)\ dm \\ &= \lim_{t \to 0} \int_M f(g_o g_H(t)\ m(t))\ P(m)\ dm. \end{aligned}$$

Reasoning as in the proof of Lemma 1, we see that

$$F_o(g) = \int_M f(g_o m_o)\ P_o(m)\ dm. \tag{13}$$

Thus, in particular, the limit defining $F_o$ exists and defines a function that is constant on cosets of $G_H$. Furthermore, since the strongly diagonal operators are left invariant, the function

$$F_t(g) = F(\delta(t)\ g)$$

is diagonally harmonic for all $t > 0$. The system of strongly diagonal operators has an elliptic operator in its span. Hence, the limit defining $F_o$ converges in the $C^\infty$ topology and $F_o$ is diagonally harmonic. It follows from formula (13) that the boundary function for $F_o$ is $f \mid G_o$, proving our lemma.

*From this point on, $P_o$ represents any Poisson kernel function for the diagonally harmonic functions on $G_o$, not just the $P_o$ defined by formula* (11).

Since $f$ is constant on cosets of $S$ in $G$, we may reduce the integral in formula (12) to an integral over $M$. Specifically, let $k = m_H h$, where $h \in N_H$ and $m_H \in M_H$. Then

$$f(km_o) = f(m_H h m_o) = f(m_H h m_o h^{-1}) = f(m_H [h, m_o]\, m_o),$$

where

$$[h, m_o] = h m_o h^{-1} m_o^{-1}.$$

But $[h, m_o] \in M_H$. Hence, changing variables in (12), yields

$$\begin{aligned} F(e) &= \int_{M_o} \int_{N_H} \int_{M_H} f(m_H m_o)\, P_H(m_H [h, m_o]^{-1}\, h)\, P_o(m_o)\, dm_o\, dh\, dm_H \\ &= \int_{M_o} \int_{M_H} f(m_H m_o)\, Q(m_H m_o)\, P_o(m_o)\, dm_o\, dm_H, \end{aligned}$$

where

$$Q(m_H m_o) = \int_{N_H} P_H(m_H m_o h m_o^{-1})\, dh. \tag{15}$$

The above computations may be summarized in the following theorem.

Theorem 3.3. *Let $Q$ be defined as in formula* (15), *where $P_H$ is the Poisson kernel function for $HJK_1$ on $G_H$. Let $P_o$ be a Poisson kernel function for the strongly diagonal HJK system on $G_o$. Then the function $P$ on $G$ defined by*

$$P(m) = Q(m)\, P_o(m_o) \tag{16}$$

*is a Poisson kernel function for the diagonally harmonic functions on $G$.*

At first glance, it might appear that the integral in formula (15) would be difficult to evaluate. Actually, there is a trick that evaluates it quite simply. Consider first the special case where we are inducing from the cone data $(S^n, M^n, I, \rho^n)$ defined in Example (1.1) relative to the canonical representation. In this case, we obtain $(S^{n+1}, M^{n+1}, I, \rho^{n+1})$. The Poisson kernel functions for the corresponding domains are unique and well known. It follows from formula (16) that

$$Q^{n+1}(m) = \frac{P^{n+1}(m)}{P^n(m_o)},$$

where $P^n$ and $P^{n+1}$ are the Poisson kernel functions for the domains defined by the cones $\mathscr{V}^n$ and $\mathscr{V}^{n+1}$, respectively, and $Q^{n+1}$ the function corresponding to $Q$ on $M^{n+1}$.

The computation of $Q$ in the general case may be reduced to that just done using the induced representation $(\tau, T)$ described in Section 2. Specifically, we noted in Section 2 that the mapping $T \times \tau$ restricts to an isomorphism $v$ of $G_H$ onto $G_H^{n+1}$. Furthermore, at the Lie algebra level, this mapping is (up to a scalar) an isometry. It follows that the $HJK_1$ operator on $G_H^n$ is a scalar multiple of the image of the corresponding operator on $G_H$. In particular,

$$P_H = P_H^n \circ v,$$

where $P_H^n$ is the Poisson kernel function for $HJK_1$ on $G_H^n$.

It follows easily from this and formula (15) that

$$\frac{P(m)}{P_o(m_o)} = Q(m) = Q^{n+1}(T(m)) = \frac{P^{n+1}(T(m))}{P^n(T_o(m_o))}. \tag{17}$$

The reader should note the similarity between this formula and formula (8). In fact, it is known [KF, p. 181] that for the domain defined by the cone in Example (1.1),

$$P^n(m) = \pi^{-n(n+1)/2} \, |D^n(I + im)|^2.$$

Using formulas (17) and (8) and mathematical induction, we prove the following theorem, which is our first major result.

THEOREM 3.4. *Let* $P(m) = \pi^{-n} \, |D(c+im)|^2$, *where* $n$ *is the dimension of* $M$. *Suppose that* $F$ *is a bounded* $C^\infty$ *function on* $\mathscr{D}$ *which is annihilated by* $HJK$ *and has continuous boundary function* $f$ *on* $M$. *Then*

$$F(ic) = \int_M f(m) \, P(m) \, dm.$$

*Proof.* We may assume by induction that

$$P_o(m_o) = \pi^{-n_o} \, |D_o(c + im_o)|^2,$$

where $n_o$ is the dimension of $M_o$. Then $n = n_o + k + 1$, where $T_o$ acts on $\mathbf{R}^k$. Then

$$\begin{aligned}P(m) &= \frac{P^{k+1}(T(m))}{P^k(T_o(m_o))}\, P_o(m_o)\\ &= \pi^{-(k+1)}\, \frac{|D^{k+1}(I + iT(m))|^2}{|D^k(I + iT_o(m_o))|^2}\, |D_o(c_o + im_o)|^2\\ &= \pi^{-n}\, |D(c + im)|^2.\end{aligned}$$

(We used formula (8) in the last equality.) This proves the theorem.

Of course, once we can compute $F(ic)$ from $f$, we can compute $F(z)$ for any $z \in \mathscr{D}$. Let $z = x + iy \in \mathscr{D}$. Write $z = g(ic)$ where $g = x \times s$. Then

$$\begin{aligned}F(z) &= \int_M f(xsm)\, P(m)\, dm\\ &= \int_M f(xsms^{-1})\, P(m)\, dm\\ &= \int_M f(x + \rho(s)\, m)\, P(m)\, dm\\ &= \int_M f(m)\, P(\rho(s)^{-1}\, (m - x))\, \det(\rho(s)^{-1})\, dm. \qquad (18)\end{aligned}$$

The Poisson kernel is, then, the function

$$\begin{aligned}P(z, m) &= P(\rho(s)^{-1}\, (m - x))\, \det(\rho(s)^{-1})\\ &= \pi^{-n}\, |D(ic + \rho(s)^{-1}\, (m - x))|^2\, (\det(\rho(s)^{-2})\, \det \rho(s))\\ &= \pi^{-n}\, |D(iy + m - x)|^2 / D(y).\end{aligned}$$

From [KF, p. 181], this is exactly the Cauchy–Szegö Poisson kerned for $\mathscr{D}$. It is known that this function is harmonic for the Laplace–Beltrami operator if and only if $\mathscr{D}$ is symmetric. On the other hand, if the boundary value functions for the Hua harmonic functions (or, more generally, the diagonally harmonic functions) are dense in $L^\infty(M)$, then this kernel would have to be harmonic in $z$. Thus, we arrive at the following theorem (note that the density of the boundary values is known in the semi-simple case by [JK]):

THEOREM 3.5. *The space of boundary functions for the Hua harmonic functions on* $\mathscr{D}$ *is dense in* $L^\infty(M)$ *if and only if* $\mathscr{D}$ *is symmetric.*

## 4. $L^2$ BOUNDARY VALUES

In this section, we continue the notation from the previous sections. We let $\mathscr{H}(\mathscr{D})$ denote the space of bounded Hua-harmonic functions on $\mathscr{D}$. We will generally denote the elements of $\mathscr{H}(\mathscr{D})$ by upper case Roman letters and their boundary functions on $M$ by the corresponding lower case Roman letter. We define

$$\mathscr{H}^2_o(\mathscr{D}) = \{F \in \mathscr{H}(\mathscr{D}) \mid f \in L^2(M)\}$$

where we use Lebesgue measure on $M$. Since the mapping $F \to f$ is one-to-one, we may put a norm on $\mathscr{H}^2_o(\mathscr{D})$ by declaring

$$\|F\| = \|f\|_2$$

We define $\mathscr{H}^2(\mathscr{D})$ to be the completion of $\mathscr{H}^2_o(\mathscr{D})$ in this norm. This space is identifiable with a closed subspace $B^2(M)$ of $L^2(M)$.

We note that from Theorem 3.4,

$$1 = \int_M P(m)\, dm$$

In particular, $P$ is in $L^1(M)$. It also follows from formula (6) that $P \in L^\infty(M)$. Thus, $P \in L^2(M)$. It follows that the Poisson integral defines a continuous mapping of $L^2(M)$ into a space of continuous functions on $\mathscr{D}$. The ellipticity of the Hua system then tells us that $\mathscr{H}^2(\mathscr{D})$ is also identifiable with a space of $C^\infty$ functions on $\mathscr{D}$. We refer to $B^2(M)$ as the space of boundary values of elements of $\mathscr{H}^2(\mathscr{D})$. Our goal is to describe $B^2(M)$.

We begin with the observation that $\mathscr{H}^2(\mathscr{D})$ is invariant under the action of $G$ on $\mathscr{D}$. In fact, the argument leading up to formula (18) shows that if $F$ is diagonally harmonic, then the boundary function for $z \to F(g^{-1}z)$ is $\pi(g)\, f$, where

$$\pi(g)\, f(m) = f(\rho(s^{-1})(m - x)),$$

which will be in $L^2(M)$ if $f$ is. In fact, the representation

$$\pi_o(g) = \det(\rho(s))^{-1/2}\, \pi(g)$$

is unitary on $L^2(M)$.

We will describe $B^2(M)$ by describing the irreducible decomposition of the restriction of $\pi_o$ to this subspace. The irreducible decomposition of $\pi_o$ itself is easily described. For $f \in L^2(M)$ and $\lambda \in M^*$, let

$$f^\wedge(\lambda) = \int_M f(x)\, e^{-i\langle x, \lambda\rangle}\, dx.$$

It is easily computed that $f \to f^\wedge$ intertwines $\pi_o$ and the representation $\tilde{\pi}_o$ on $L^2(M^*)$ defined by

$$\tilde{\pi}_o(x \times s)\, h(\lambda) = \det(\rho(s)^{1/2})\, e^{-i\langle x, \lambda\rangle} h(\rho^*(s^{-1})\, \lambda). \tag{19}$$

Since $\rho^*$ has open orbits (e.g., $\mathscr{V}^*$), we know that the union of the open orbits is dense in $M^*$. Let $\mathscr{V}_i$, for $i = 1, 2, ..., k$, be the set of open $\rho^*$ orbits. Let $\mathscr{L}_i$ be the space of functions in $L^2(M^*)$ that are supported in $\mathscr{V}_i$. Then the $\mathscr{L}_i$ are closed, invariant subspaces for $\tilde{\pi}_o$.

Lemma 4.1. *The restriction $\pi_i$ of $\tilde{\pi}_o$ to $\mathscr{L}_i$ is irreducible and the $\pi_i$ are mutually inequivalent.*

*Proof.* For each $i$, let $\lambda_i$ be a fixed base point in $\mathscr{V}_i$ and let $\chi_i$ be the character of $M$ defined by

$$\chi_i(m) = e^{i\langle m, \lambda_i\rangle}.$$

It is easily seen that $\pi_i$ is equivalent with the representation of $G$ induced from $\chi_i$. From Mackey theory, the irreducibility of $\pi_i$ is equivalent with proving that the stabilizer of $\chi_i$ under the adjoint action of $G$ on $M$ is just $M$ itself. This, in turn, will follow if we can show that the $\rho^*$ stabilizer of $\lambda_i$ is trivial. However, since the $r^*$ orbit of $\lambda_i$ is open, this stabilizer must have zero dimension. The triviality follows from the complete solvability of $S$.

To prove the mutual inequivalence of the $\pi_i$, it suffices to show that the restrictions of these representations to $M$ are inequivalent, which is clear from formula (19).

It follows from Lemma 4.1 that there is a set $i_1, i_2, ..., i_k$ of indices for which

$$(B^2(M))^\wedge = \bigoplus \sum_j \mathscr{L}_{i_j}.$$

This decomposition defines the irreducible decomposition of $\pi_o$. The following theorem follows from these comments. We call this the Paley–Wiener theorem because it characterizes the boundary values of the space of $L^2$ harmonic functions in terms of the support of their Fourier transforms.

Paley–Wiener Theorem. *Let $f \in L^2(M)$. Then $f \in B^2(M)$ if and only if the Fourier transformation $f^\wedge$ is supported in the union of the $\mathscr{V}_{i_j}$.*

The orbits $\mathscr{V} i_j$ are called *harmonic*. If we can characterize the harmonic orbits, then we have an essentially complete picture of the boundary values of the elements of $\mathscr{H}^2(\mathscr{D})$. At first glance, it appears that obtaining such a characterization should not be difficult. Let $P^\wedge(z, \cdot)$ be as defined at the end of Section 1. If $f \in \mathscr{L}_i$, then $f^\wedge$ is supported in $\mathscr{V}_i$. Thus

$$\begin{aligned} F(z) &= \int_M f(m)\, P(z, m)\, dm \\ &= \int_{M^*} f^\wedge(\lambda)\, P^\wedge(z, \lambda)\, dl \\ &= \int_{\mathscr{V}_i^*} f^\wedge(\lambda)\, P^\wedge(z, \lambda)\, dl \end{aligned}$$

This defines a Hua-harmonic function for all $f$ in $\mathscr{L}_i$ if and only if

$$z \to P^\wedge(z, \lambda)$$

is Hua-harmonic for all $\lambda \in \mathscr{V}_i^*$. Actually, it is easily seen that for all $s \in S$

$$P^\wedge(\rho(s)\, z, \lambda) = P^\wedge(z, \rho^*(s)\, \lambda)$$

Thus, the orbit will be harmonic if $P^\wedge$ is harmonic for any single $\lambda$ in the orbit. Hence, we arrive at the following theorem.

THEOREM 4.2. *The orbit $\mathscr{V}_i$ is harmonic if and only if there is a $\lambda_i \in \mathscr{V}_i$ such that $z \to P^\wedge(z, \lambda_i)$ is harmonic for the Hua system.*

Unfortunately, there does not seem to be a simple formula for $P^\wedge$. Hence, we do not yet have a simple criterion for the harmonicity of a given orbit. This, hopefully, will be the subject of further work on this problem.

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